New Product-Type and Ratio-Type Exponential Estimators of the Population Mean Using Auxiliary Information in Sample Surveys

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Abstract

This paper addresses the problem of estimating the population mean of the study variable \( y \) using information on an auxiliary variable \( x \). A class of Exponential-Type Estimators has been suggested along with its properties under large sample approximation. It is identified that the usual Unbiased Estimator, Product-Type and Ratio-Type Exponential Estimators are members of the proposed class of Exponential-Type Estimators. It has been shown that the proposed class of Exponential-Type Estimators is more efficient than the usual Unbiased Estimator and some existing Estimators. An empirical study is carried out in support of the present study.

Keywords

Auxiliary variable, Study variable, Bias, Mean squared error

1. Introduction

It is common to use the auxiliary information at the estimation stage in order to obtain improved estimates of the population mean \( \bar{y} \) of the study variate \( y \). Out of many, Ratio, Product and Regression methods are good examples in this context. When the auxiliary variable \( x \) is positively (high) correlated with the study variable \( y \), the Ratio method of estimation is quite effective. On the other hand, if this correlation is negative, the Product method of estimation can be employed.

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Consider a finite population \( U = (u_1, u_2, \ldots, u_N) \) of size \( N \). Let \((y, x)\) be the study and auxiliary variables, respectively, taking values \((y_i, x_i)\) on the \( i^{th} \) unit \( U_i (i = 1, 2, \ldots, N) \) of the population. Let \((\bar{Y}, \bar{X})\) be the population means of \((y, x)\), respectively. It is assumed that the population mean \( \bar{X} \) of the auxiliary variable \( x \) is known. For estimating the population mean \( \bar{Y} \) of the study variable \( y \), a simple random sample of size \( n \) is selected without replacement from the population \( U \).

Then the Classical Ratio Estimator for the population mean \( \bar{Y} \) is defined by

\[
t_R = \hat{R} \bar{X}
\]

where,

\[
\hat{R} = \frac{\bar{Y}}{\bar{X}}, \bar{X} \neq 0 \text{ is the estimate of the Ratio } R \text{ of the population means.}
\]

\[
\bar{y} = \left( \frac{1}{n} \right) \sum_{i=1}^{n} y_i \text{ and } \bar{x} = \left( \frac{1}{n} \right) \sum_{i=1}^{n} x_i \text{ are the un-weighted sample means of } y \text{ and } x, \text{ respectively. This Estimator is only efficient if the variable } y \text{ and } x \text{ are strongly positively correlated.}
\]

The ordinary Product Estimator for \( \bar{Y} \) is defined by

\[
t_p = \frac{\hat{P}}{\bar{X}}
\]

where, \( \hat{P} = \bar{y} \bar{x} \) is the estimate of the Product \( P \) of the population means, will often be used if the two variables are supposed to be strongly negatively correlated.

It is to be noted that the Estimator \( t_p \) is due to Robson (1957) and revisited by Murthy (1964). Bahl and Tuteja (1991) suggested Ratio and Product-Type Exponential Estimators for the population mean \( \bar{Y} \) respectively as

\[
t_{Re} = \bar{y} \exp\left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right)
\]

and

\[
t_{Pe} = \bar{y} \exp\left( \frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}} \right)
\]

The simple Expansion Estimator for the population mean \( \bar{Y} \) is

\[
t_0 = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i
\]
is used otherwise.

Murthy (1967, p. 370) pointed out that the variability of the sample mean $\bar{x}$ is usually less than that of sample mean $\bar{y}$. If $C(\bar{x})$ denotes the coefficient of variation of $\bar{x}$ likewise $C(\bar{y})$ that of $\bar{y}$, then

$$C^2(\bar{x}) = \frac{(1-f)}{n} C_x^2$$ and

$$C^2(\bar{y}) = \frac{(1-f)}{n} C_y^2$$

where,

$f = n/N$ is the sampling fraction, $C_x = S_x/\bar{X}$ and $C_y = S_y/\bar{Y}$ are the population coefficients of variation for the two variables.

$$S_x^2 = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})^2$$ and

$$S_y^2 = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2.$$

It follows that if $C_x = a C_y$, we have,

$$C(\bar{x}) = a C(\bar{y}); \quad 0 < a \leq 1$$

We suppose that the observation on $y$ and $x$ are all non-negative, so that the sample and population means are all positive. Murthy (1964) suggested the use of

$$t_R \quad \text{if} \quad \frac{\rho}{a} > \frac{1}{2}, \quad \quad (1.7)$$

$$t_r \quad \text{if} \quad -\frac{1}{2} \leq \frac{\rho}{a} \leq \frac{1}{2}, \quad \quad (1.8)$$

$$t_p \quad \text{if} \quad \frac{\rho}{a} < -\frac{1}{2}, \quad \quad (1.9)$$

where,

$$\rho = \frac{S_{xy}}{S_x S_y}$$

is the correlation coefficient between the study variable $y$ and the auxiliary variable $x$ and $S_{xy} = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})(y_i - \bar{Y})$.

Here, of course

$$\frac{\rho}{a} \geq -\frac{1}{a}$$

and

$$\frac{\rho}{a} \leq \frac{1}{a} \quad \text{as} \quad |\rho| \leq 1$$

In this paper motivated by Sahai (1979), we have suggested a variant of the Product and Ratio-Type Exponential Estimators, with the intention to improve
their efficiency. We have obtained the Bias and Mean Squared Error (MSE) of the Proposed Estimator to the first degree of approximation and compared with those of the well known methods Simple Expansion, Ratio and Product. An empirical study is given in support of the present study.

2. The Suggested Estimator

Motivated by Sahai (1979), we derive the following modified Exponential-Type Estimator for the population mean $\bar{Y}$ as

$$t_{Me} = \bar{y} \exp \left\{ \frac{(\bar{x} + \theta \overline{X}) - (X + \theta \overline{X})}{(\bar{x} + \theta \overline{X}) + (X + \theta \overline{X})} \right\}$$

$$= \bar{y} \exp \left\{ \frac{\pi (1 - \theta) + (\theta - 1)\overline{X}}{\pi (1 + \theta) + (\theta + 1)\overline{X}} \right\}$$

$$= \bar{y} \exp \left\{ \frac{(\theta - 1)\overline{X} - (\theta - 1)\overline{x}}{(\theta + 1)\overline{X} + (\theta + 1)\overline{x}} \right\}$$

$$t_{Me} = \bar{y} \exp \left\{ \frac{(\theta - 1)(\overline{X} - \overline{x})}{(\theta + 1)(\overline{X} + \overline{x})} \right\}$$

(2.1)

where, $\theta$ is a scalar used as a design parameter. It is worth mention that, for $\theta = 1$ $t_{Me} = \bar{y}$ and that for $\theta = 0$, $t_{Me} = t_{Re}$.

Moreover, if $\theta$ is very large $t_{Me}$ is almost the same as the $t_{Re}$

i.e. \[ \lim_{\theta \to \infty} t_{Me} = \lim_{\theta \to \infty} \bar{y} \exp \left\{ \frac{(\theta - 1)(\overline{X} - \overline{x})}{(\theta + 1)(\overline{X} + \overline{x})} \right\} \]

\[ \approx t_{Re} = \bar{y} \exp \left\{ \frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}} \right\} . \]

Sometimes, a good guess of the value of $\rho/\alpha$ is available from a pilot sample past data, experience or otherwise. In other practical situations, the value of $\rho/\alpha$ may be known or guessed to be in certain interval. Using such knowledge, one can give a suitable value to $\theta$, the design parameter in order that the Proposed Modified Estimator will have a smaller Mean Squared Error (MSE) than the usual Ratio, Product, Exponential-Type Ratio, Exponential-Type Product and Simple Expansion Estimator, respectively.
2.1 Sampling Bias and Mean Squared Error of the Estimators: Following Murthy (1967), we write,
\[ \bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1) \]
such that
\[ E(e_0) = E(e_1) = 0 \quad \text{and} \quad E(e_0^2) = \frac{(1 - f)}{n} C_x^2, \quad E(e_1^2) = \frac{(1 - f)}{n} \rho C_x C_y. \]
We can reasonably suppose that the sample size is large to make \(|e_0| < 1\) and \(|e_1| < 1\).

Further, to validate the first order large sample approximation we are going to obtain, we suppose that the sample size is large enough to obtain \(|e_0|\) and \(|e_1|\) small so that the terms involving \(e_0\) and / or \(e_1\) in a degree greater than two will be negligible; an assumption which is usually not unrealistic.

Now expressing eq. (2.1) in terms of \(e_0\) and \(e_1\), we have,
\[ t_{\text{Mc}} = \bar{Y}(1 + e_0) \exp \left\{ -\frac{(\theta - 1) e_1}{(\theta + 1)(2 + e_1)} \right\} \]
\[ = \bar{Y}(1 + e_0) \exp \left\{ \frac{(1 - \theta) e_1}{(1 + \theta)2} \left( 1 + \frac{e_1}{2} \right)^{-1} \right\} \]
\[ = \bar{Y}(1 + e_0) \exp \left\{ \frac{G e_1}{2} \left( 1 + \frac{e_1}{2} \right)^{-1} \right\} \quad (2.1.1) \]
where, \( G = \frac{(1 - \theta)}{(1 + \theta)}. \)

Expanding the right hand side of eq. (2.1.1), multiplying and neglecting terms of \(e\)'s having power greater than two, we have,
\[ t_{\text{Mc}} \approx \bar{Y} \left[ 1 + e_0 + \frac{G e_1}{2} + \frac{G e_0 e_1}{2} + \frac{G(G - 2)}{8} e_1^2 \right] \]
and
\[ (t_{\text{Mc}} - \bar{Y}) \approx \bar{Y} \left[ e_0 + \frac{G e_1}{2} + \frac{G e_0 e_1}{2} + \frac{G(G - 2)}{8} e_1^2 \right] \quad (2.1.2) \]
Taking expectation on both sides of eq. (2.1.2), we get the Bias of the first degree of approximation as
\[ B(t_{Me}) = \bar{Y} \left( 1 - f \right) \left[ \frac{G \rho C_y C_x}{2} + \frac{G(G-2)C_x^2}{8} \right] = B_0 \left( \frac{G}{2} \right) \left[ \rho + \frac{(G-2)}{4a} \right] \] (2.1.3)

where, \( B_0 = \frac{a(1-f)\bar{Y}C_y^2}{n} \) and \( G = \frac{(1-\theta)}{(1+\theta)} \).

Squaring both sides of eq. (2.1.2) and neglecting terms of e's having power greater than two, we have,

\[ (t_{Me} - \bar{Y})^2 \approx \bar{Y}^2 \left[ e_0^2 + \frac{G^2 e_1^2}{4} + Ge_0 e_1 \right]. \] (2.1.4)

Taking expectation of both sides of eq. (2.1.4), we get the MSE of \( t_{Me} \) to the first degree of approximation as

\[ \text{MSE}(t_{Me}) = \text{MSE}(t_{Me}) = \left( 1 - f \right) n \left[ \frac{C_y^2 + G \rho C_y C_x + \left( G^2 C_x^2 \right)}{4} \right] \]

\[ = \frac{(1-\theta)}{(1+\theta)} \left[ 1 + \frac{G^2 a^2}{4} + aG \rho \right] \] (2.1.5)

where,

\[ V_0 = \frac{(1-f)}{n} S_y^2 = \text{Var}(\bar{Y}) \] (2.1.6)

The MSE \( t_{Me} \) is minimum when

\[ G_0 = -2 \left( \frac{\rho}{a} \right) \] (2.1.7)

Thus, the resulting minimum MSE of \( t_{Me} \) is given by

\[ \text{min MSE}(t_{Me}) = V_0 (1 - \rho^2) \] (2.1.8)

which equals to the approximate variance/MSE of the usual Regression Estimator \( \bar{Y}_n = \bar{Y} + \hat{\beta}(\bar{X} - \bar{X}) \) (2.1.9)

2. Efficiency Comparison

3.1 When the Scalar \( G(\text{or} \theta) \) Coincides with its True Optimum Value: It is well known under Simple Random Sampling WithOut Replacement (SRSWOR) scheme that
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\[ \text{Var}(\bar{y}) = \text{MSE}(\bar{y}) = \left( \frac{1-f}{n} \right) S_y^2 = \left( \frac{1-f}{n} \right) \bar{Y}^2 C_y^2 = V_0 \]  

(3.1.1)

To the first degree of approximation, Biases and Mean Squared Errors of the Estimators, \( t_R, t_P, t_{Re} \) and \( t_{Pe} \) are respectively given by

\[ B(t_R) = B_0 (a - \rho) \]  

(3.1.2)

\[ B(t_P) = B_0 \rho \]  

(3.1.3)

\[ B(t_{Re}) = \left( \frac{B_0}{8} \right) (3a - 4\rho) \]  

(3.1.4)

\[ B(t_{Pe}) = \left( \frac{B_0}{8} \right) (4\rho - a) \]  

(3.1.5)

\[ \text{MSE}(t_R) = V_0 \left( 1 + a^2 - 2\rho \alpha \right) \]  

(3.1.6)

\[ \text{MSE}(t_P) = V_0 \left( 1 + a^2 + 2\rho \alpha \right) \]  

(3.1.7)

\[ \text{MSE}(t_{Re}) = V_0 \left[ 1 + \left( \frac{a}{4} \right) (a - 4\rho) \right] \]  

3.1.8

\[ \text{MSE}(t_{Pe}) = V_0 \left[ 1 + \left( \frac{a}{4} \right) (a + 4\rho) \right] \]  

(3.1.9)

where,

\[ B_0 = \frac{a(1-f)\bar{Y}C_y^2}{n} \]

From eq. (2.1.8), (3.1.1), (3.1.5), (3.1.6), (3.1.7) and (3.1.8), we have,

\[ \text{MSE}(\bar{y}) - \min \ \text{MSE}(t_{Me}) = V_0 \rho^2 \geq 0 \]  

(3.1.10)

\[ \text{MSE}(t_R) - \min \ \text{MSE}(t_{Me}) = V_0 (a - \rho)^2 \geq 0 \]  

(3.1.11)

\[ \text{MSE}(t_P) - \min \ \text{MSE}(t_{Me}) = V_0 (a + \rho)^2 \geq 0 \]  

(3.1.12)

\[ \text{MSE}(t_{Re}) - \min \ \text{MSE}(t_{Me}) = V_0 \left( \frac{a}{2} - \rho \right)^2 \geq 0 \]  

(3.1.13)

\[ \text{MSE}(t_{Pe}) - \min \ \text{MSE}(t_{Me}) = V_0 \left( \frac{a}{2} + \rho \right)^2 \geq 0 \]  

(3.1.14)

It is observed from eq. (3.1.10) to eq. (3.1.14) that the Proposed class of Estimators \( t_{Me} \) is more efficient than

- The usual Unbiased Estimator \( \bar{y} \) unless the correlation between the study variable \( y \) and the auxiliary variable \( x \) is zero. We note that when \( \rho = 0 \)
(i.e. the two variables \( y \) and \( x \) are uncorrelated) both the Estimators \( \bar{y} \) and \( t_{Me} \) are equally efficient.

- The usual Ratio Estimator \( t_R \) except when \( a = \rho \), the case where both the Estimators \( t_R \) and \( t_{Me} \) are equally efficient.
- The usual Product Estimator \( t_P \) except when \( a = -\rho \), the case where both the Estimators \( t_P \) and \( t_{Me} \) are equally efficient.
- The Bahl and Tuteja (1991) Ratio-Type Exponential Estimator \( t_{Re} \) except when \( a = 2\rho \), the case where both the Estimators \( t_{Re} \) and \( t_{Me} \) are equally efficient.
- The Bahl and Tuteja (1991) Product-Type Exponential Estimator \( t_{Pe} \) except when \( a = -2\rho \), the case where both the Estimators \( t_{Pe} \) and \( t_{Me} \) are equally efficient.

3.2 When the Scalar \( G(\text{or} \theta) \) does not Coincide with Its True Optimum Value:

In consequence of formula eq. (2.6), we have,

\[
MSE(t_{Me}) - Var(\bar{y}) = V_0a^2\left(\frac{G^2}{4} + \frac{G\rho}{a}\right)
= V_0a^2\left(\frac{G^2 - GG_0}{2}\right) \quad \therefore G_0 = -2\left(\frac{\rho}{a}\right)
= V_0a^2\left(G^2 - 2GG_0\right)
= V_0a^2\left(G^2 - 2GG_0 + G_0^2 - G_0^2\right)
= V_0a^2\left\{\left(G - G_0\right)^2 - G_0^2\right\}
\]

which is less than ‘zero’ if \( \left(G - G_0\right)^2 < G_0^2 \)
i.e. \( \left|G - G_0\right| < \left|G_0\right| \) \quad (3.2.1)

or equivalently \( \min\left\{0, -\frac{4\rho}{a}\right\} < G < \max\left\{0, -\frac{4\rho}{a}\right\} \)

From eq. (2.1.5) and eq. (3.1.6), we have,
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\[ \text{MSE}(t_{Me}) - \text{MSE}(t_R) = V_0 a^2 \left( \frac{G^2}{4} + \frac{G \rho}{a} - 1 + \frac{2 \rho}{a} \right) \]

\[ = V_0 a^2 \left( \frac{G^2}{4} - \frac{GG_0}{2} - 1 - G_0 \right) \therefore G_0 = -2 \left( \frac{\rho}{a} \right) \]

\[ = \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 - 4 - 4G_0 \right) \]

\[ = \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 + G_0^2 - 4 - 4G_0 - G_0^2 \right) \]

\[ = \frac{V_0 a^2}{4} \left( (G - G_0)^2 - (2 + G_0)^2 \right) \]

which is less than ‘zero’ if \((G - G_0)^2 < (2 + G_0)^2\)

i.e. \(|G - G_0| < |2 + G_0|\) \hspace{1cm} (3.2.2)

or equivalently \(\min \left\{ -2, 2 \left( \frac{1 - \frac{2 \rho}{a}}{a} \right) \right\} < G < \max \left\{ -2, 2 \left( 1 - \frac{2 \rho}{a} \right) \right\} \)

From eq. (2.1.5) and eq. (3.1.7), we have,

\[ \text{MSE}(t_{Me}) - \text{MSE}(t_p) = V_0 a^2 \left( \frac{G^2}{4} + \frac{G \rho}{a} - 1 - \frac{2 \rho}{a} \right) \]

\[ = V_0 a^2 \left( \frac{G^2}{4} - \frac{GG_0}{2} + 1 + G_0 \right) \therefore G_0 = -2 \left( \frac{\rho}{a} \right) \]

\[ = \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 + 4G_0 \right) \]

\[ = \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 + G_0^2 + 4G_0 - G_0^2 \right) \]

\[ = \frac{V_0 a^2}{4} \left( (G - G_0)^2 - (2 - G_0)^2 \right) \]

which is less than ‘zero’ if \((G - G_0)^2 < (2 - G_0)^2\)

i.e. \(|G - G_0| < |2 - G_0|\) \hspace{1cm} (3.2.3)

or equivalently \(\min \left\{ 2, 2 \left( 1 + \frac{2 \rho}{a} \right) \right\} < G < \max \left\{ 2, -2 \left( 1 + \frac{2 \rho}{a} \right) \right\} \)

From eq. (2.1.5) and eq. (3.1.8), we have,
\[MSE(t_{Me}) - MSE(t_{Re}) = V_0 a^2 \left( \frac{G^2}{4} + \frac{G\rho}{a} - \frac{1}{4} + \frac{\rho}{a} \right)\]
\[= V_0 a^2 \left( \frac{G^2}{4} - \frac{GG_0}{2} - \frac{1}{4} - \frac{G_0}{2} \right); \quad G_0 = -2 \left( \frac{\rho}{a} \right)\]
\[= \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 - 1 - 2G_0 \right)\]
\[= \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 + G_0^2 - 1 - 2G_0 - G_0^2 \right)\]
\[= \frac{V_0 a^2}{4} \left( G - G_0 \right)^2 - (1 + G_0)^2\]

which is less than 'zero' if \((G - G_0)^2 < (1 + G_0)^2\)

i.e. \[|G - G_0| < |1 + G_0|\] (3.2.4)

or equivalently \[
\min \left\{ -1, \left( 1 - \frac{4\rho}{a} \right) \right\} < G < \max \left\{ -1, \left( 1 - \frac{4\rho}{a} \right) \right\}
\]

From eq. (2.1.5) and eq. (3.1.9), we have,
\[MSE(t_{Me}) - MSE(t_{Re}) = V_0 a^2 \left( \frac{G^2}{4} + \frac{G\rho}{a} - \frac{1}{4} + \frac{\rho}{a} \right)\]
\[= V_0 a^2 \left( \frac{G^2}{4} - \frac{GG_0}{2} - \frac{1}{4} + \frac{G_0}{2} \right); \quad G_0 = -2 \left( \frac{\rho}{a} \right)\]
\[= \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 - 1 + 2G_0 \right)\]
\[= \frac{V_0 a^2}{4} \left( G^2 - 2GG_0 + G_0^2 - 1 + 2G_0 - G_0^2 \right)\]
\[= \frac{V_0 a^2}{4} \left( G - G_0 \right)^2 - (1 - G_0)^2\]

which is less than ‘zero’ if \((G - G_0)^2 < (1 - G_0)^2\)

i.e. \[|G - G_0| < |1 - G_0|\] (3.2.5)

or equivalently \[
\min \left\{ 1 + \left( \frac{4\rho}{a} \right) \right\} < G < \max \left\{ 1 + \left( \frac{4\rho}{a} \right) \right\}
\]
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It follows from eq. (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.5) that the Proposed Modified Exponential Estimator \( t_\text{Me} \) is more efficient than

- The usual Unbiased Estimator \( \bar{y} \) if \( |G - G_0| < |G_0| \) or equivalently
  \[
  \min \left( 0, -\frac{4\rho}{a} \right) < G < \max \left( 0, -\frac{4\rho}{a} \right)
  \]
- The usual Ratio Estimator \( t_R \) if \( |G - G_0| < |2 + G_0| \) or equivalently
  \[
  \min \left\{ -2, 2 \left( 1 - \frac{2\rho}{a} \right) \right\} < G < \max \left\{ -2, 2 \left( 1 - \frac{2\rho}{a} \right) \right\}
  \]
- The usual Product Estimator \( t_P \) if \( |G - G_0| < |2 - G_0| \) or equivalently
  \[
  \min \left\{ 2, -2 \left( 1 + \frac{2\rho}{a} \right) \right\} < G < \max \left\{ 2, -2 \left( 1 + \frac{2\rho}{a} \right) \right\}
  \]
- The Ratio-Type Exponential Estimator \( t_{Re} \) due to Bahl and Tuteja (1991) if \( |G - G_0| < |1 + G_0| \) or equivalently
  \[
  \min \left\{ -1, \left( 1 - \frac{4\rho}{a} \right) \right\} < G < \max \left\{ -1, \left( 1 - \frac{4\rho}{a} \right) \right\}
  \]
- The Product-Type Exponential Estimator \( t_{Pe} \) due to Bahl and Tuteja (1991) if \( |G - G_0| < |1 - G_0| \) or equivalently
  \[
  \min \left\{ 1, -\left( 1 + \frac{4\rho}{a} \right) \right\} < G < \max \left\{ 1, -\left( 1 + \frac{4\rho}{a} \right) \right\}
  \]

4. Accuracy of First Order Approximations to MSE’s

We have already compared the MSEs of the Proposed Estimator and the other Estimators, subject to the first order of approximations. Here, we intend to examine the accuracy of these approximations by obtaining the second order approximations to the MSEs. We assume that \( c_x = c_y = C \), say \( a = 1 \) and that the sample comes from a large Bivariate Normal population as otherwise very complicated expressions are obtained, see Sahai (1979, p.33). For this case, we have,
\[
\mu_{30} = \mu_{03} = \mu_{12} = \mu_{21} = 0
\]
\[
\mu_{04} = \mu_{40} = 3C^4
\]
\[ \mu_{31} = \mu_{13} = 3\rho C^4 \]
\[ \mu_{22} = (1 + 2\rho^2)C^4 \]

where,
\[
E(e_i^i e_j^j) \approx \frac{\mu_{ij}}{n^b}, \quad i, j = 0, 1, 2, 4 \quad \text{and} \quad b = 2 \quad \text{for} \quad i + j = 4
\]

Thus, to the second order of approximations, we have,
\[
t_{Me} = \bar{Y}(1 + e_0)\exp\left\{\frac{Ge_1}{2} \left(1 + \frac{e_1}{2}\right)^{-1}\right\}
\]
\[
= \bar{Y}(1 + e_0)\left[1 + \frac{Ge_1}{2} + \frac{G(G - 2)e_1^2}{8} + \frac{G(G^2 - 6G + 6)e_1^3}{48} + \frac{G(G^3 - 12G^2 + 36G - 24)e_1^4}{384}\right]
\]

We denote by \( a_1 = \frac{G}{2}, a_2 = \frac{G(G - 2)}{8}, a_3 = \frac{G(G^2 - 6G + 6)}{48} \) and \( a_4 = \frac{G(G^3 - 12G^2 + 36G - 24)}{384} \).

then
\[
t_{Me} = \bar{Y}(1 + e_0)\left[1 + a_1 e_1 + a_2 e_1^2 + a_3 e_1^3 + a_4 e_1^4 + \ldots\right]
\]
\[
= \bar{Y}\left[1 + e_0 + a_1 e_1 + a_2 e_0 e_1 + a_3 e_1^2 + a_4 e_0 e_1 + a_3 e_1^3 + a_4 e_0 e_1 + a_4 e_1^4 + \ldots\right]
\]

Neglecting term of \( e \)'s having power greater than four, we have,
\[
t_{Me} \approx \bar{Y}[1 + e_0 + a_1 (e_1 + e_0 e_1) + a_2 (e_1^2 + e_0 e_1^2) + a_3 (e_1^3 + e_0 e_1^3) + a_4 e_1^4]
\]

or
\[
(t_{Me} - \bar{Y}) \approx \bar{Y}[e_0 + a_1 (e_1 + e_0 e_1) + a_2 (e_1^2 + e_0 e_1^2) + a_3 (e_1^3 + e_0 e_1^3) + a_4 e_1^4]
\]

Taking expectation of both sides of eq. (4.1), we get the bias of \( t_{Me} \) as
\[
B(t_{Me}) = \frac{3C^2}{n} \left[ (\rho a_1 + a_2) + \frac{3C^2}{n} (a_3 \rho + a_4) \right]
\]

(4.2)

Squaring both sides of eq. (4.1) and neglecting term of \( e \)'s having power greater than two, we have,
\[
(t_{Me} - \bar{Y})^2 = \bar{Y}^2 \left[ e_0^2 + 2a_1 e_0 e_1 + a_1^2 e_1^2 + 2(a_1^2 + a_2) e_0 e_1^2 + 2a_1 a_2 e_1^2 e_0 + 2a_2 a_3 e_1^3 + 2a_2 a_3 e_1^3 e_0 + (a_2^2 + 2a_1 a_2) e_1^4 \right]
\]

(4.3)
Taking expectation of both sides of eq. (4.3), we get the Mean Squared Error to the second order of approximation (retaining the term up to fourth degree in $e_0$ and/or $e_1$), we have,

$$\text{MSE}(t_{me}_i) = \text{MSE}(t_{me}_i) + \left(\frac{C^4Y^2}{n^2}\right)\left[\left(a_1^2 + 2a_2\right)\left(1 + 2\rho^2\right) + 6(a_3 + 2a_4)\rho + 3(a_5^2 + 2a_6)\right]$$

$$= \text{MSE}(t_{me}_i) + \frac{C^4Y^2}{64n^2} \left[7G^4 + 4G^3(14\rho - 9) + 4G^2(16\rho^2 - 36\rho + 17) - 16G(4\rho^2 - 3\rho + 2)\right]$$

(4.4)

In case a good guess of $G_0$ is available, $G \equiv G_0 = -2\rho$ (with $a = 1$). Then putting $G = -2\rho$ in eq. (4.4), we have,

$$\text{MSE}(t_{me}_i) = \text{MSE}(t_{me}_i) + \frac{C^4Y^2}{4n^2} \rho \left(4 + 11\rho - 10\rho^2 - 5\rho^3\right)$$

(4.5)

$$= \text{MSE}(t_{me}_i) + \frac{C^2Y^2}{4n} \rho \left(4 + 15\rho + 5\rho^2\right) \left(1 + \rho\right)$$

(4.6)

$$= \left(\frac{C^2Y^2}{n}\right) \left((1 - \rho^2) + \frac{C^2}{4n} \rho (4 + 15\rho + 5\rho^2)\right)$$

(4.7)

Here, we note that

$$\text{MSE}(t_{me}_i) = \text{MSE}(Y_{ir})_i = \frac{C^2Y^2}{n} \left(1 - \rho^2\right)$$

(4.8)

Inserting $G = -1$ and $G = 1$ in eq. (4.4), we get the MSE expressions for Ratio-Type and Product-Type Exponential Estimators to the second order of approximation, respectively as

$$\text{MSE}(t_{re}_i) = \text{MSE}(t_{re}_i) + \frac{C^2Y^2}{16n} \left(143 - 248\rho + 128\rho^2\right)$$

(4.9)

$$= \frac{C^2Y^2}{n} \left[\left(\frac{5}{4} - \rho\right) + \frac{C^2}{64n} (143 - 248\rho + 128\rho^2)\right]$$

And

$$\text{MSE}(t_{pe}_i) = \text{MSE}(t_{pe}_i) + \frac{C^2}{16n} \left(7 - 40\rho\right)$$

(4.10)

$$= \frac{C^2Y^2}{n} \left[\left(\frac{5}{4} + \rho\right) + \frac{C^2}{64n} (7 - 40\rho)\right]$$

where,
\[ MSE(t_{Re})_I = \frac{C^2\overline{Y}^2}{4n}(5 - 4\rho) \] (4.11)

and

\[ MSE(t_{Pe})_I = \frac{C^2\overline{Y}^2}{4n}(5 + 4\rho) \] (4.12)

From eq. (4.6) and eq. (4.9), we have,

\[ MSE(t_{Re})_II - MSE(t_{Me})_II = \frac{C^2\overline{Y}^2}{n} \left[ \left( \frac{1}{2} - \rho \right)^2 + \frac{C^2}{64n} (143 - 312\rho - 48\rho^2 + 160\rho^3 + 80\rho^4) \right] \]

which is positive if

\[ \left[ \left( \frac{1}{2} - \rho \right)^2 + \frac{C^2}{64n} (143 - 312\rho - 48\rho^2 + 160\rho^3 + 80\rho^4) \right] > 0 \] (4.13)

Further, from eq. (4.7) and eq. (4.10), we have,

\[ MSE(t_{Pe})_II - MSE(t_{Me})_II = \frac{C^2\overline{Y}^2}{n} \left[ \left( \frac{1}{2} + \rho \right)^2 + \frac{C^2}{64n} (7 - 104\rho - 176\rho^2 + 160\rho^3 + 80\rho^4) \right] \]

which is positive if

\[ \left[ \left( \frac{1}{2} + \rho \right)^2 + \frac{C^2}{64n} (7 - 104\rho - 176\rho^2 + 160\rho^3 + 80\rho^4) \right] > 0 \] (4.14)

To compare \( t_{Me} \) with regression Estimator \( \overline{y}_r \), we have from eq. (2.9) and eq. (4.7) that for large sample the former are approximately as efficient as the latter, if a good guess \( G_0 \) is available. In case the sample (not very large) is drawn from a large Bivariate Normal population, the Mean Squared Error of the usual Regression Estimator \( \overline{y}_r \); to the second order approximation(i.e. terms up to \( n^{-2} \)) \([\text{Cochran (1967), 7.4, p.198, } \gamma_1 = 0, \gamma_2 = 0 \text{ for Normal Distribution}] \) given as

\[ MSE(\overline{y}_r)_II = C^2\overline{Y}^2 \left( 1 - \rho^2 \right) \left( \frac{1}{n} + \frac{1}{n^2} \right) \] (4.15)

Also, for this case, if a good guess of \( G_0 \) is available, we have from eq. (4.6) and eq. (4.15)

\[ MSE(\overline{y}_r)_II - MSE(t_{Me})_II = \frac{C^2\overline{Y}^2}{n^2} \left[ (1 - \rho^2) - \frac{C^2}{4} \rho(1 - \rho)(4 + 15\rho + 5\rho^2) \right] > 0 \]
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if \[ \left(1 - \rho^2\right) - \frac{C^2}{4} \rho (4 + 15 \rho + 5 \rho^2) > 0 \] (4.16)

Thus the supposition \( C_x = C_y = C \); say \( a = 1 \) and that the sample comes from a
large Bivariate Normal population, the Mean Squared Error of the usual Ratio
Estimator \( t_R = y\left(\frac{X}{X}\right) \) and the Product Estimator \( t_P = y\left(\frac{X}{X}\right) \) to the second
degree of approximation are respectively given by

\[
MSE(t_R)_n = MSE(t_R)_l \left[1 + \frac{C^2}{n} (6 - 3 \rho)\right]
\]

\[
= \left[\frac{C^2 Y^2}{n} \left(2(1 - \rho) + 6\frac{C^2}{n} (2 - 3 \rho + \rho^2)\right)\right] \quad (4.17)
\]

and

\[
MSE(t_P)_n = MSE(t_P)_l \left[1 + \frac{C^2}{2n} (1 + \rho^2)\right]
\]

\[
= \left[\frac{C^2 Y^2}{n} \left(2(1 + \rho) + \frac{C^2}{n} (1 + \rho^2)\right)\right] \quad (4.18)
\]

From eq. (4.9) and eq. (4.17), we have,

\[
MSE(t_R)_n - MSE(t_R)_l = \left[\frac{C^2 Y^2}{n} \left[\left(\frac{3}{4} - \rho\right) + \frac{C^2}{n} \left(6(1 - \rho)(2 - \rho) - \frac{1}{64} (143 - 248 \rho + 128 \rho^2)\right)\right]\right]
\]

which is greater than ‘zero’ if

\[
\left[\left(\frac{3}{4} - \rho\right) + \frac{C^2}{n} \left(6(1 - \rho)(2 - \rho) - \frac{1}{64} (143 - 248 \rho + 128 \rho^2)\right)\right] > 0 \quad (4.19)
\]

Next, from eq. (4.9) and eq. (4.18), we have,

\[
MSE(t_P)_n - MSE(t_P)_l = \left[\frac{C^2 Y^2}{n} \left[\left(\frac{3}{4} + \rho\right) + \frac{C^2}{64n} (57 + 168 \rho)\right]\right]
\]

which is greater than ‘zero’ if

\[
\left[\left(\frac{3}{4} + \rho\right) + \frac{C^2}{64n} (57 + 168 \rho)\right] > 0 \quad (4.20)
\]
Thus the Ratio-Type $\left( t_{re} \right)$ and Product-Type $\left( t_{pe} \right)$ Exponential Estimators are better than the usual Ratio $\left( t_r \right)$ and Product $\left( t_p \right)$ Estimators, respectively, as long as the conditions eq. (4.19) and eq. (4.20) are satisfied.

5. Empirical Study

To examine the merits of the suggested Estimator we have considered five natural population data sets. The descriptions of the population are given below

**Population 1:** [Murthy (1967)]
- $y$: Output,
- $x$: Fixed Capital,
- $N = 80, n = 20$, $\bar{y} = 51.8264$, $\bar{x} = 11.2646$, $C_y = 0.3542$,
- $C_x = 0.7507$, $\rho = 0.9413$, $C = 0.4441$, $f = 0.25$.

**Population 2:** [Murthy (1967)]
- $y$: Output,
- $x$: Number of workers,
- $N = 80, n = 20$, $\bar{y} = 51.8264$, $\bar{x} = 2.8513$, $C_y = 0.3542$,
- $C_x = 0.9484$, $\rho = 0.9150$, $C = 0.3417$, $f = 0.25$.

**Population 3:** [Das (1988)]
- $y$: Number of agricultural laborers for 1971,
- $x$: Number of agricultural laborers for 1961,
- $N = 278, n = 30$, $\bar{y} = 39.0680$, $\bar{x} = 25.1110$, $C_y = 1.4451$,
- $C_x = 1.6198$, $\rho = 0.7213$, $C = 0.6435$, $f = 0.1079$.

**Population 4:** [Steel and Torrie (1960)]
- $y$: Log of leaf burn in secs,
- $x$: Chlorine percentage,
- $N = 30, n = 6$, $\bar{y} = 0.6860$, $\bar{x} = 0.8077$, $C_y = 0.700123$,
- $C_x = 0.7493$, $\rho = -0.4996$, $C = -0.3203$, $f = 0.20$. 
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**Population 5:** [Maddala (1977)]

\[ y : \text{Consumption per capita}, \]
\[ x : \text{Deflated price of veal}, \]
\[ N = 16, n = 4, \bar{Y} = 7.6375, \bar{X} = 75.4313, C_y = 0.2278, \]
\[ C_x = 0.0986, \rho = -0.6823, C = -1.5761, f = 0.25. \]

We have computed the range of \( G \) and percent relative efficiencies of different Estimators \( (\bar{y}, t_R, t_P, t_{Re}, t_{Pe} \text{ and } t_{Me}) \) of the population mean \( \bar{Y} \) with respect to \( \bar{y} \). Findings are compiled in Tables 1, 2 and 3.

It is observed from Table 1, 2 and 3 that there is enough scope of selecting the value of scalar \( G(\text{or}\theta) \) in order to get Estimators better than sample mean \( \bar{y}, t_R, t_P, t_{Re} \text{ and } t_{Pe} \). We also note that the Proposed class of Estimators \( t_{Me} \) is better than Conventional Estimators \( \bar{y} \) even if the scalar \( G(\text{or}\theta) \) slider away from the true optimum value of \( G(\text{or}\theta) \). Larger gain in efficiency is observed if the scalar \( G(\text{or}\theta) \) moves around the vicinity of the true optimum value.

**Acknowledgements**

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**Table 1:** Range of \( G \) in which Proposed Estimator \( t_{Me} \) is better than \( \bar{y}, t_R, t_P, t_{Re} \text{ and } t_{Pe} \).

<table>
<thead>
<tr>
<th>Population</th>
<th>Range of ( G ) in which Proposed Estimator ( t_{Me} ) is better than ( \bar{y}, t_R, t_P, t_{Re} \text{ and } t_{Pe} )</th>
<th>Common range of ( G ) in which ( t_{Me} ) is better than ( \bar{y}, t_R, t_P, ) ( t_{Re} \text{ and } t_{Pe} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( (-1.7765,0) ) ( (-2.02235) ) ( (-3.7765,2) ) ( (-1,-0.7765) ) ( (-2.7765,1) ) ( (-1,-0.7765) )</td>
<td>( (-1,-0.7765) )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( (-1.3669,0) ) ( (-2.06331) ) ( (-3.3669,2) ) ( (-1.3669) ) ( (-2.3669,1) ) ( (-1.3669) )</td>
<td>( (-1.3669) )</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( (-2.5742,0) ) ( (-2.05742) ) ( (-4.5742,2) ) ( (-1.5742,1) ) ( (-3.5742,1) ) ( (-1.5742,0.5740) )</td>
<td>( (-1.5742,0.5740) )</td>
</tr>
<tr>
<td>( 4 )</td>
<td>( (0.18673) ) ( (-2.38673) ) ( (-0.13282) ) ( (-1.38673) ) ( (0.8673,1) ) ( (0.8673,1) )</td>
<td>( (0.8673,1) )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( (0.63059) ) ( (-2.83059) ) ( (2.43054) ) ( (-1.73059) ) ( (1.53059) ) ( (-1.43054) )</td>
<td>( (-1.43054) )</td>
</tr>
</tbody>
</table>
Table 2: PREs of Estimators $\bar{y}$, $t_R$, $t_p$, $t_{Re}$ and $t_{Pe}$ with respect to $\bar{y}$.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\text{PRE}(\bullet, \bar{y})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Population</td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>100.0000 100.0000 100.0000 100.0000 100.0000</td>
</tr>
<tr>
<td>$t_R$</td>
<td>66.5810 30.5860 156.3967 31.1061 56.2431</td>
</tr>
<tr>
<td>$t_p$</td>
<td>10.5463 7.6514 25.8171 92.9342 167.5887</td>
</tr>
<tr>
<td>$t_{Re}$</td>
<td>781.3982 292.0779 197.7846 54.9135 74.5067</td>
</tr>
<tr>
<td>$t_{Pe}$</td>
<td>24.2836 19.0754 47.1121 133.0386 133.0649</td>
</tr>
</tbody>
</table>

Table 3: PREs of $t_{Me}$ with respect to $\bar{y}$ for different values of $G$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\theta$</th>
<th>$\text{PRE}(t_{Me}, \bar{y})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Population</td>
</tr>
<tr>
<td>-2.0000</td>
<td>-3.0000</td>
<td>66.5810 30.5860 156.3967 31.1061 56.2431</td>
</tr>
<tr>
<td>-1.7500</td>
<td>-3.6667</td>
<td>105.4984 45.4209 182.7966 35.5534 60.2317</td>
</tr>
<tr>
<td>-1.5000</td>
<td>-5.0000</td>
<td>187.3984 73.4677 202.4393 40.8775 64.5841</td>
</tr>
<tr>
<td>-1.2500</td>
<td>-9.0000</td>
<td>383.2829 135.4864 208.2654 47.2636 69.3319</td>
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<tr>
<td>-1.0000</td>
<td>0.0000</td>
<td>781.3982 292.0779 197.7846 54.9135 74.5067</td>
</tr>
<tr>
<td>-0.7500</td>
<td>7.0000</td>
<td>738.4366 585.7795 175.3442 64.0167 80.1386</td>
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<tr>
<td>-0.5000</td>
<td>3.0000</td>
<td>353.0569 448.2367 148.3074 74.6863 86.2537</td>
</tr>
<tr>
<td>-0.2500</td>
<td>1.6667</td>
<td>174.9985 200.1900 122.3232 86.8380 92.8714</td>
</tr>
<tr>
<td>0.0000</td>
<td>1.0000</td>
<td>100.0000 100.0000 100.0000 100.0000 100.0000</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.6000</td>
<td>63.7373 57.9872 81.8494 113.0935 107.6315</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.3333</td>
<td>43.6933 37.4100 67.4411 124.3406 115.7344</td>
</tr>
<tr>
<td>0.7500</td>
<td>0.1429</td>
<td>31.9699 26.0031 56.0853 131.5695 124.2463</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.0000</td>
<td>24.2836 19.0754 47.1121 133.0386 133.0649</td>
</tr>
<tr>
<td>1.2500</td>
<td>-0.1111</td>
<td>19.0533 14.5708 39.9776 128.5994 142.0399</td>
</tr>
<tr>
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<td>-0.2000</td>
<td>15.3392 11.4840 34.2528 118.7285 150.9669</td>
</tr>
<tr>
<td>1.7500</td>
<td>-0.2727</td>
<td>12.6097 9.2794 29.6138 106.2421 159.5862</td>
</tr>
<tr>
<td>2.0000</td>
<td>-0.3333</td>
<td>10.5463 7.6514 25.8171 92.9342 167.5887</td>
</tr>
</tbody>
</table>
References