A SYSTEM OF LINEAR QUATERNION MATRIX EQUATIONS WITH APPLICATIONS

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(Received: January 1, 2006)

ABSTRACT: The general solution of the system of quaternion matrix equations

\[ A_1 X = C_1, \quad XB_2 = C_2, \quad A_3 X = C_3, \quad XB_4 = C_4 \]

is considered. A necessary and sufficient, condition for the existence and for the representation of the general solution of such a system is given. As applications, necessary and sufficient conditions for the existence and the expressions of the bisymmetric solutions for the matrix equation \( XB = C \); is given centrosymmetric solution of

\[
\begin{cases}
A_1 X = C_1 \\
XB_2 = C_2
\end{cases}
\]

and the persymmetric solution to

\[
\begin{cases}
A_1 X = C_1 \\
A_2 X = C_3
\end{cases}
\]

are given, respectively. Moreover some auxiliary results on other systems of quaternion matrix equations are also mentioned.

Keywords: System of linear quaternion matrix equations; reflexive inverse of a matrix; bisymmetric matrix; centrosymmetric, persymmetric matrix

2000 AMS Subject Classifications 15A24, 15A33, 15A57, 15A09

1. INTRODUCTION

In 1976, Khatri and Mitra [1] have studied the Hermitian solutions to the following matrix equations over the complex field

\[ AX = C \] \hspace{1cm} (1.1)

and

\[
\begin{cases}
A_1 X = C_1 \\
XB_2 = C_2
\end{cases}
\] \hspace{1cm} (1.2)

The inverse problem of the matrix equation (1.1), i.e. the matrix equation

\[ X = A^{-1} C \] \hspace{1cm} (1.3)

1Supported by PhD Foundation of Shandong Finance University
has been investigated in [2]-[4]. The symmetric solution and Hermitian solutions for (1.1) have been investigated by many authors, e.g. Vetter [5], Magnus and Neudecker [6], Don [8], and Dai [9]. Mitra in [10] presented further investigation for the system (1.2). Chu [7] studied the symmetric solutions of the system of linear real matrix equations.

\[
\begin{cases}
A_1X = C_1 \\
A_2X = C_2
\end{cases}
\]  

(1.4)

Controsymmetric, persymmetric and bisymmetric matrices have been widely discussed (e.g [11]-[12]), which are very useful in various physics and engineering problems [12], in the study of some markov processes [14], in the numerical solution of certain differential equations, information theory, linear system theory, linear estimation theory [13], and others.

Motivated by the work mentioned above, we consider the following system

\[
\begin{cases}
A_1X = C_1 \\
XB_2 = C_2 \\
A_3X = C_3 \\
XB_4 = C_4
\end{cases}
\]  

(1.5)

over the real quaternion field \( \mathbb{H} \). In section 2, we derive a necessary and sufficient condition for the existence and representation of the general solution for the system (1.5). Moreover 1 as special cases, the corresponding results on the systems (1.4) and

\[
\begin{cases}
XB_2 = C_2 \\
XB_4 = C_4
\end{cases}
\]  

(1.6)

over \( \mathbb{H} \) are also given. In section 3, as applications of Section 2, we present necessary and sufficient conditions for the existence of bisymmetric solution for the matrix equation (1.3), persymmetric solution of the systems (1.4), centrosymmetric solution for the system (1.2) over \( \mathbb{H} \); and give expression of such solutions when the corresponding conditions hold.

Throughout this we denote the set of all \( m \times n \) matrices over \( \mathbb{H} \) by \( \mathbb{H}^{m \times n} \), the identity matrix with appropriate sizes by \( I \), a reflexive inverse of a matrix \( A \) over \( \mathbb{H} \) by \( A^r \). This last one satisfies simultaneously \( AA^r A = A \) and \( A^r AA^r = A^r \). Moreover, \( L_A = I - A^r A, \quad R_A = I - AA^r \) where \( A^r \) is any but fixed.
2. THE GENERAL SOLUTION TO THE SYSTEM (1.5) OVER $\mathbb{H}$

In this section we consider the system (1.5) over $\mathbb{H}$.

Lemma 2.1. (Lemma 2.2 in [23]) Let $A \in H^{m \times n}$, $B \in H^{r \times s}$ and $C \in H^{m \times s}$.

Then the following condition are equivalent:

1. The matrix equation
   \[ A X B = C \]  
   (2.1)

is consistent.


3. $C L_B = 0$, $R_A C = 0$

In that case, the general solution of (2.1) can be expressed as

\[ X = A^+ C B^+ B + L_A V + U R_B \]  
(2.2)

where $U$ and $V$ are any matrices with compatible dimensions over $\mathbb{H}$.

Lemma 2.2 Let $A_1 \in H^{m \times n}$, $B_2 \in H^{r \times s}$, $C_1 \in H^{m \times r}$, $C_2 \in H^{s \times s}$ be known and $X \in H^{n \times r}$ unknown. Then the system (1.2) is consistent if and only if

\[ A_1 A_1^+ C_1 = C_2 B_2^+ B_3 = C_2, A_1 C_2 = C_1 B_2 \]  
(2.3)

in that case, the general solution of the system (1.2) is

\[ X = A_1^+ C_1 + L_{A_1} C_2 B_2^+ + L_{A_1} Y R_{B_2}, \]  
(2.4)

where $Y$ is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions.

Now we give the main result of this paper.

Theorem 2.3: Let $A_i \in H^{m \times n}$, $A_i \in H^{k \times n}$, $B_2 \in H^{r \times s}$, $B_4 \in H^{r \times r}$, $C_1 \in H^{m \times r}$, $C_2 \in H^{s \times s}$, $C_3 \in H^{k \times s}$, $C_4 \in H^{n \times n}$ be known matrices and $X \in H^{n \times r}$ unknown; and $K = A_3 L_{A_3}, M = L_{A_3} L_{K}, N = R_{B_4} B_4, T = C_3 - A_3 A_3^+ C_1, Q_i = T - K C_2 B_2^+, Q = C_4 - A_4 A_4^+ C_3 B_4 - L_{A_3} C_2 B_2^+ B_4 - L_{A_4} K^+ T N.$

Then the system (1.5) is consistent if and only if

\[ K K^+ T R_{B_4} = Q_i L_{Q_i} N = 0, R_{B_4} Q_i = 0, \]  
(2.7)

\[ A_i A_i^+ C_i, i = 1,3, A_4 C_2 = C_1 B_2, C_4 B_4^+ B_4 = C_1, j = 2,4 \]  
(2.8)

In this case, the general solution of (1.5) can be expressed as the following:

\[ X = A_1^+ C_1 + L_{A_3} C_2 B_2^+ + L_{A_4} K^+ T R_{B_4} + Q_i N + R_{B_4} + M Z R_{N} R_{B_4} \]  
(2.9)

where $Z$ is an arbitrary matrix over $\mathbb{H}$ with compatible dimensions.

Proof. The Proof consists of two main parts. We first show that the matrix $X$ that has the form of (2.9) is a solution of the system (1.5) under the
assumption (2.7) and (2.8), then prove that any solution of the system (1.5) can be expressed as the form of (2.9), when (2.7) and (2.8) hold.

Suppose that (2.7) and (2.8) hold. Noting that $A_4L_{A_4} = 0$, $MM^+Q = Q$, we can easily verify that the matrix $X$ that has the form of (2.9) is a solution of the equation $A_4X = C_1$. It follows from (2.8) and $R_{B_2}B_2 = 0$ that $XB_2 = 0$.

Note that $A_3M = A_3L_{A_3}L_{K} = KL_{K} = 0$. So $MM^+Q = Q$ yields $A_4Q = 0$. Hence by (2.5) and the first equation of (2.7), we have the following

$$A_4X = A_4A_4^+C_1 + KC_2B_2^+ + Q = C_3.$$ 

Now we prove that $XB_4 = C_4$. By $R_{N}N = 0$, $QN^+N = Q$ and (2.6), we have

$$XB_4 = A_4^+C_1B_4 + L_{A_4}C_2B_2^+B_4 + L_{A_4}K^+TN + Q$$

$$= C_4$$

To sum up, a matrix $X$ that has the form of (2.9) is a solution of the system (1.5) under the assumptions (2.7) and (2.8).

Conversely, assume that the system (1.5) has a solution $X_0$, then $X_0$ is a solution of the system (1.2) and $A_3X_0 = C_3, X_0B_4 = C_4$. By Lemma 2.2 and Lemma 2.1, (2.8) holds and

$$X_0 = A_4^+C_1 + L_{A_4}C_2B_2^+ + L_{A_4}YR_{B_4},$$

(2.10)

So

$$C_3 = A_4X_0 = A_4(A_4^+C_1 + L_{A_4}C_2B_2^+ + L_{A_4}YR_{B_4}),$$

i.e., the matrix equation $KYR_{B_1} = Q_1$ is consistent for $Y$ where $Q_1$ is defined by (2.5). Note that $R_{B_1}$ is idempotent and a reflexive inverse of $R_{B_2}$ is itself and $R_{B_2} = I - R_{B_2}$. Hence it follows from Lemma 2.1 that the first equation of (2.7) holds and

$$Y = K^+Q_1R_{B_2} + L_{K}V + U(I - R_{B_2})$$

where $U$ and $V$ are arbitrary matrices with appropriate sizes over $H$.

Therefore $B_2^+R_{B_2} = 0$ and (2.5), (2.10) becomes

$$X_0 = A_4^+C_1 + L_{A_4}C_2B_2^+ + L_{A_4}K^+TR_{B_2} + L_{A_4}L_{K}VR_{B_2}. $$

Hence by (2.6), (2.5) and $C_4 = X_0B_4$, we have the fact that
MVN = Q.

So Lemma 2.1 yields the later two equations of (2.7).

Now we show that \( X_0 \) can be expressed as the form of (2.9). It follows from

\[
L_{A_1}X_0 = X_0 - A_1^+ C_1
\]

that

\[
L_KX_0R_{B_2} = X_0R_{B_2} - K^+ KX_0R_{B_2}
= X_0 - X_0B_2^+B_2^+ - K^+ KX_0R_{B_2}
= X_0 - C_2B_2^+ - K^+ A_4L_{A_4}X_0R_{B_2}
= X_0 - C_2B_2^+ - K + C_3R_{B_2} + K^+ A_4A_1^+C_1R_{B_2}
\]

Therefore by (2.5), we have the following

\[
L_{A_4}L_KX_0R_{B_2} = X_0 - A_1^+ C_1 - L_{A_4}C_2B_2^+ - L_{A_4}K^+ C_3R_{B_2} + L_{A_4}K^+ A_4A_1 + C_1R_{B_2}
= X_0 - A_1^+ C_1 - L_{A_4}C_2B_2^+ - L_{A_4}K^+TR_{B_2}
\]

whence in view of (2.6),

\[
L_{A_4}L_KX_0NN^+R_{B_2} = L_{A_4}L_KX_0R_{B_2}B_4N^+R_{B_2}
= (X_0 - A_1^+ C_1 - L_{A_4}C_2B_2^+ - L_{A_4}K^+TR_{B_2})B_4N^+R_{B_2}
= (C_4 - A_1^+ C_1B_4 - L_{A_4}C_2B_2^+B_4 - L_{A_4}K^+TN)N^+R_{B_2}
= QN^+R_{B_2}.
\]

Hence

\[
MX_0R_{R_2} = L_{A_4}L_KX_0R_{R_2}
= L_{A_4}L_KX_0R_{R_2} - L_{A_4}L_KX_0NN^+R_{B_2}
= X_0 - A_1^+ C_1 - L_{A_4}C_2B_2^+ - L_{A_4}K^+TR_{B_2} - QN^+R_{B_2},
\]

i.e.

\[
X_0 = A_1^+ C_1 + L_{A_4}C_2B_2^+ + L_{A_4}K^+TR_{B_2} + QN^+R_{B_2} + MX_0R_{R_2}.
\]

So \( X_0 \) can be expressed as the form of (2.9) where \( Z = X_0 \).

In Theorem 2.3, let \( B_2, B_4, C_2 \) and \( C_4 \) vanish. Then we have the following.
Corollary 2.4 Let \( A_i \in \mathbb{H}^{m \times n}, A_3 \in \mathbb{H}^{k \times n}, C_1 \in \mathbb{H}^{m \times r}, C_3 \in \mathbb{H}^{k \times r} \) be known matrices and \( X \in \mathbb{H}^{n \times r} \) unknown; and \( K = A_3 L_{A_i}, T = C_3 - A_i A_i^+ C_1 \). Then the system (1.4) is consistent if and only if.

\[
K K^+ T = T, A_i A_i^+ C_1 = C_1, i = 1, 3,
\]

in which case, the general solution of (1.4) can be expressed as the following:

\[
X = A_i^+ C_1 + L_{A_i} K^+ T + L_{A_i} L_K Z
\]

where \( Z \) is an arbitrary matrix over \( \mathbb{H} \) with compatible dimensions.

In Theorem 2.3, let \( A_1, C_1, A_3 \) and \( C_3 \) vanish. Then we have the following.

Corollary 2.5. Let \( B_2 \in \mathbb{H}^{r \times s}, B_4 \in \mathbb{H}^{r \times d}, C_2 \in \mathbb{H}^{n \times s}, C_4 \in \mathbb{H}^{n \times d} \) known matrices and \( X \in \mathbb{H}^{n \times r} \) unknown; and \( N = R_{B_2} B_4, Q = C_4 - C_2 B_2^+ B_4 \). Then the system (1.6) is consistent if and only if.

\[
C_j B_j^+ B_j = C_j, j = 2, 4; Q L_N = 0,
\]

in which case, the general solution of (1.6) can be expressed as the following:

\[
X = C_2 B_2^+ + Q N^+ R_{B_2} + Z R_N R_{B_2}
\]

where \( Z \) is an arbitrary matrix over \( \mathbb{H} \) with compatible dimensions.

3. THE VARIOUS SYMMETRIC SOLUTIONS TO SOME QUATERNION MATRIX EQUATIONS

In this section, we use Theorem 2.3 to consider the bisymmetric solution to the matrix equation (1.1), the centrosymmetric solution to the system (1.2) and the persymmetric solutions to the systems (1.4). We denote the \( n \times n \) permutation matrix whose elements along the southwest northeast diagonal are ones and whose remaining elements are zeros by \( V_n \). By \[22\], we have the following definition.

Definition 3.1. Suppose that

\[
A = (a_{ij}) \in \mathbb{H}^{m \times n}, A^* = (a_{ji}) \in \mathbb{H}^{n \times m}, A^* = (\overline{a_{ji}}) \in \mathbb{H}^{n \times m}, A^\# = (a_{n-i, n-j+1}) \in \mathbb{H}^{m \times n},
\]

where \( a_{ji} \) is the conjugate of the quaternion \( a_{ji} \),
(i) The matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is called symmetric if \( A = A^\ast \).

(ii) The matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is called persymmetric if \( A = A^{(*)} \).

(iii) The matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is called centrosymmetric if

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\]

i.e. \( a_{ij} = a_{n+1-i,n+1-j} \).

(iv) The matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is called bisymmetric if

\[
a_{ij} = a_{n+1-i,n+1-j} = a_{n+1-i,n+1-j}.
\]

Remarks 3.1. (1) Of the three matrix properties --- symmetric, persymmetric and centrosymmetric --- any two imply the third one. So a bisymmetric matrix \( A \) implies \( A = A^\ast = A^{(*)} = A^\# \).

(2) \( V_n = V_n^\ast = V_n^\# = V_n^\# \).

(3) For matrices \( A \) and \( B \) over \( \mathbb{H} \), it is easy to verify that

\[
(AB)^\ast = A^\ast B^\ast, (AB)^{(*)} = B^{(*)} A^{(*)}, (AB)^\# = A^\# B^\#,
\]

\[
(A^\ast)^\ast = (A^\ast)^\ast, (A^{(*)})^{(*)} = (A^{(*)})^{(*)}, (A^\#)^\ast = (A^\#)^\ast.
\]

3.1. The Bisymmetric Solution to (1.3).

Now we use Theorem 2.3 to give a necessary and sufficient condition for the matrix equation (1.3) to have a bisymmetric solution and the expression of such a solution.

Theorem 3.1. Let \( B, C \in \mathbb{H}^{m \times n} \) be known and \( X \in \mathbb{H}^{m \times n} \) unknown; and

\[
K = B^{(*)} L_{B^\#}, N = R_B B^\#, M = L_{B^\#} L_K, T = C^{(*)} - B^{(*)} (B^\#)^\# C^\#, Q_1 = T - KCB^+, Q = C^\# - (CB^+)^\# B^\# - L_{B^\#} C B^+ B^\# - L_B K^\# T N.
\]

Then the matrix equation (1.3) has a bisymmetric solution if and only if

\[
K K^\# T R_B = Q_1, Q L_N = 0, R_B Q = 0, B^+ B C = C, C^+ B = B^+ C,
\]

in which case, the general bisymmetric solution of (1.3) can be expressed as the following:

\[
X = \frac{1}{4} \left( X_1 + X_1^* + X_1^{(*)} + X_1^\# \right)
\]

with

\[
X_1 = (CB^+)^* + L_{B^+} C B^+ + L_{B^+} K^+ T R_B + Q N^+ R_B + M Z R_N R_B
\]

where \( Z \) is an arbitrary matrix over \( \mathbb{H} \) with compatible dimension.
Proof. It is sufficient to show that the matrix equation (1.3) has a bisymmetric solution is equivalent to the following system

\[
\begin{align*}
B^*X = & C^* \\
XB = & C \\
B^{(*)}X = & C^{(*)} \\
XB'' = & C''
\end{align*}
\]  

(3.2)

is consistent for \( X \). In fact, if the matrix equation (1.3) has a bisymmetric solution \( X \), then \( X \) is obviously a solution of (3.2). Conversely, assume that (3.2) has a solution \( X_1 \), then \( X_1^*B = C, X_1^{(*)}B = C, X_1''B = C \). Hence it is easy to verify that (3.1) is a bisymmetric solution of the matrix equation (1.3). Theorem 2.3 yields the remaining part of the proof.

Remark 3.2. (1) Similarly, we can investigate the bisymmetric solution to the quaternion matrix equation (1.1).

(2) Using Theorem 3.1, we can study the bisymmetric solution to the system (1.2), (1.4) and (1.6). In fact, the system (1.2), (1.4) and (1.6) has a bisymmetric solution is equivalent to the following system

\[
\begin{align*}
\begin{bmatrix}
A_1 \\
B_2^*
\end{bmatrix} X = & \begin{bmatrix}
C_1 \\
C_2^*
\end{bmatrix} \\
\begin{bmatrix}
A_1^* \\
A_3
\end{bmatrix} X = & \begin{bmatrix}
C_1^* \\
C_3
\end{bmatrix} \\
\begin{bmatrix}
B_2^* \\
B_4^*
\end{bmatrix} X = & \begin{bmatrix}
C_2^* \\
C_4^*
\end{bmatrix}
\end{align*}
\]  

has a bisymmetric solution, respectively.

3.2. The Centrosymmetric Solution to the system (1.2).

Now we use Theorem 2.3 to consider the centrosymmetric solution to the system (1.2).

Theorem 3.2.

Let \( A_i \in \mathbb{H}^{m \times n}, B_2 \in \mathbb{H}^{r \times s}, C_1 \in \mathbb{H}^{m \times r}, C_2 \in \mathbb{H}^{n \times s} \) be unknown matrices and \( X \in \mathbb{H}^{n \times r} \) unknown; and

\[
K = A_1^bL_A, M = L_A^rL_K, N = R_{B_2}B_2'', T = C_1'' - A_1''A_1 + C_1, Q_1 = T - K C_2B_2^+, 
\]
\[ Q = C_2^\# - A_i^* C_i B_2^\# - L_{A_i} C_2 B_2 + L_{A_i} K^* TN. \]

Then the system (1.2) has a centrosymmetric solution if and only if

\[ KK^* TR_{B_1} = Q_1, QL_N = 0, R_M Q = 0, A_i A_i^* C_i = C_i, A_i C_2 = C_i B_2, C_2 B_2^* B_2 = C_2, \]

in which case, the general centrosymmetric solution of (1.2) can be expressed as the following:

\[ X = \frac{1}{2} (X_1 + X_1^\#) \quad (3.3) \]

with

\[ X_1 = A_i^* C_i + L_{A_i} C_2 B_2 + L_{A_i} K^* TR_{B_2} + QN^* R_{B_2} + MZ R_{B_2}, \]

where \( Z \) is an arbitrary matrix over \( \mathbb{H} \) with compatible dimensions.

**Proof.** It is sufficient to show that the system (1.2) has a centrosymmetric solution equivalent to the system

\[
\begin{cases}
A_i X = C_i \\
X B_2 = C_2 \\
A_i^* X = C_i^\#
\end{cases}
\quad (3.4)
\]

is consistent for \( X \). As a matter of fact, if the system (1.2) has a centrosymmetric solution \( X \), then \( X \) is obviously a solution of the system (3.4). Conversely, suppose that the system (3.4) has a solution \( X_1 \). Noting that \( A_i X_1^\# = C_i, X_1^\# B_2 = C_2 \), by \( A_i^* X_1 = C_i^\# \) and \( X_1^\# B_2 = C_2^\# \). Hence it is easy to verify that (3.3) is a centrosymmetric solution of the system (1.2). The remaining parts of the proof can be obtained by Theorem 2.3.

### 3.3. The Persymmetric Solution to (1.4).

Now we consider the persymmetric solution to the system (1.4) by using Theorem 2.3.

**Theorem 3.3.**

Let \( A_i, C_i \in \mathbb{H}^{m \times n}, A_3, C_3 \in \mathbb{H}^{k \times n} \) be known matrices and

\[
K = A_3 L_A, M = L_A L, N = R_A, A_3^{(*)}, T = C_3 - A_3^* A_i^* C_i, Q_i = T - K(A_i^* C_i)^{(*)},
\]

\[
Q = C_3^{(*)} - A_i^* C_i A_3^{(*)} - L_A (A_i^* A_i^* C_i)^{(*)} - L_A K^* TN.
\]

Then the system (1.4) has a persymmetric solution if and only if
in which case, the general solution of (1.4) can be expressed as the following:

\[ X = \frac{1}{2} \left( X_1 + X_1^{(*)} \right) \quad (3.5) \]

with \( X_1 = A_1^T C_1 + L_{A_4} (A_4^T C_4)^{(*)} + L_{A_4} K^T R_{A_4}^{(*)} + Q N^T R_{A_4}^{(*)} + M Z R_{A_4}^{(*)} \), where \( Z \) is an arbitrary matrix over \( \mathbb{H} \) with compatible dimensions.

Proof. It is sufficient to show that the system (1.4) has a persymmetric solution is equivalent to the system

\[
\begin{cases}
A_1 X = C_1 \\
A_2^T X = C_2^{(*)} \\
A_3 X = C_3 \\
A_4^T X = C_4^{(*)}
\end{cases}
\quad (3.6)
\]

is consistent for \( X \). In fact, if the system (1.4) has a persymmetric solution \( X \), then \( X \) is obviously a solution of the system (3.6). Conversely, assume that the system (3.6) has a solution \( X_1 \). Noting that \( A_1 X_1^{(*)} = C_1, A_3 X_1^{(*)} = C_3 \) by \( A_4 X_1^{(*)} = C_4^{(*)} \) and \( A_4 X_1^{(*)} = C_4^{(*)} \). Hence it is easy to verify that (3.5) is a persymmetric solution of the system (1.4). The remaining parts of the proof can be obtained by Theorem 2.3.

Remark 3.3. Similarly we can investigate the persymmetric solution of the system (1.6).

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